

Generating functions for generalized Stirling type numbers, Array type polynomials, Eulerian type polynomials and their applications

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Abstract

The first aim of this paper is to construct new generating functions for the generalized λ -Stirling type numbers of the second kind, generalized array type polynomials and generalized Eulerian type polynomials and numbers, attached to Dirichlet character. We derive various functional equations and differential equations using these generating functions. The second aim is provide a novel approach to deriving identities including multiplication formulas and recurrence relations for these numbers and polynomials using these functional equations and differential equations. Furthermore, by applying p -adic Volkenborn integral and Laplace transform, we derive some new identities for the generalized λ -Stirling type numbers of the second kind, the generalized array type polynomials and the generalized Eulerian type polynomials. We also give many applications related to the class of these polynomials and numbers.

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1. INTRODUCTION, DEFINITIONS AND PRELIMINARIES

Throughout this paper, we use the following standard notations:

$\mathbb{N} = \{1, 2, 3, \dots\}$, $\mathbb{N}_0 = \{0, 1, 2, 3, \dots\} = \mathbb{N} \cup \{0\}$ and $\mathbb{Z}^- = \{-1, -2, -3, \dots\}$. Here, \mathbb{Z} denotes the set of integers, \mathbb{R} denotes the set of real numbers and \mathbb{C} denotes the set of complex numbers. We assume that $\ln(z)$ denotes the principal branch of the multi-valued function $\ln(z)$ with the imaginary part $\Im(\ln(z))$ constrained by

$$-\pi < \Im(\ln(z)) \leq \pi.$$

Furthermore,

$$0^n = \begin{cases} 1 & n = 0 \\ 0 & n \in \mathbb{N}, \end{cases}$$

$$\binom{x}{v} = \frac{x(x-1) \cdots (x-v+1)}{v!}$$

and

$$\{z\}_0 = 1 \text{ and } \{z\}_j = \prod_{d=0}^{j-1} (z-d),$$

where $j \in \mathbb{N}$ and $z \in \mathbb{C}$ cf. ([13], [29]).

The generating functions have various applications in many branches of Mathematics and Mathematical Physics. These functions are defined by linear polynomials, differential relations, globally referred to as *functional equations*. The functional equations arise in well-defined combinatorial contexts and they lead systematically to well-defined classes of functions (cf. see, for detail, [16]). Although, in the literature, one can find extensive investigations related to the generating functions for the Bernoulli, Euler and Genocchi numbers and polynomials and also their generalizations, the λ -Stirling numbers of the second kind, the array polynomials and the Eulerian polynomials, related to nonnegative real parameters, have not been studied yet. Therefore, Section 2, Section 3 and Section 4 of this paper deal with new classes of generating functions which are related to generalized λ -Stirling type numbers of the second kind, generalized array type polynomials and generalized Eulerian polynomials, respectively. By using these generating functions, we derive many functional equations and differential equations. By using these equations, we investigate and introduce fundamental properties and many new identities for the generalized λ -Stirling type numbers of the second kind, the generalized array type polynomials and the generalized Eulerian type polynomials and numbers. We also derive multiplication formulas and recurrence relations for these numbers and polynomials.

The remainder of this study is organized as follows:

In section 5, we derive new identities related to the generalized Bernoulli polynomials, the generalized Eulerian type polynomials, generalized λ -Stirling type numbers and the generalized array polynomials.

In section 6, we give relations between generalized Bernoulli polynomials and generalized array polynomials.

In section 7, We give an application of the Laplace transform to the generating functions for the generalized Bernoulli polynomials and the generalized array type polynomials.

In section 8, by using the bosonic and the fermionic p -adic integral on \mathbb{Z}_p , we find some new identities related to the Bernoulli polynomials, the generalized Eulerian type polynomials and Stirling numbers.

2. GENERATING FUNCTION FOR GENERALIZED λ -STIRLING TYPE NUMBERS OF THE SECOND KIND

The Stirling numbers are used in combinatorics, in number theory, in discrete probability distributions for finding higher order moments, etc. The Stirling number of the second kind, denoted by $S(n, k)$, is the number of ways to partition a set of n objects into k groups. These numbers occur in combinatorics and in the theory of partitions.

In this section, we construct a new generating function, related to nonnegative real parameters, for the generalized λ -Stirling type numbers of the second kind. We derive some elementary properties including recurrence relations of these numbers. The following definition provides a natural generalization and unification of the λ -Stirling numbers of the second kind:

Definition 1. Let $a, b \in \mathbb{R}^+$ ($a \neq b$), $\lambda \in \mathbb{C}$ and $v \in \mathbb{N}_0$. The generalized λ -Stirling type numbers of the second kind $\mathcal{S}(n, v; a, b; \lambda)$ are defined by means of the following generating function:

$$f_{\mathcal{S},v}(t; a, b; \lambda) = \frac{(\lambda b^t - a^t)^v}{v!} = \sum_{n=0}^{\infty} \mathcal{S}(n, v; a, b; \lambda) \frac{t^n}{n!}. \quad (2.1)$$

Remark 1. By setting $a = 1$ and $b = e$ in (2.1), we have the λ -Stirling numbers of the second kind

$$\mathcal{S}(n, v; 1, e; \lambda) = S(n, v; \lambda)$$

which are defined by means of the following generating function:

$$\frac{(\lambda e^t - 1)^v}{v!} = \sum_{n=0}^{\infty} S(n, v; \lambda) \frac{t^n}{n!},$$

cf. ([29], [46]). Substituting $\lambda = 1$ into above equation, we have the Stirling numbers of the second kind

$$S(n, v; 1) = S(n, v),$$

cf. ([13], [29], [46]). These numbers have the following well known properties:

$$S(n, 0) = \delta_{n,0},$$

$$S(n, 1) = S(n, n) = 1$$

and

$$S(n, n-1) = \binom{n}{2},$$

where $\delta_{n,0}$ denotes the Kronecker symbol (see [13], [29], [46]).

By using (2.1), we obtain the following theorem:

Theorem 1.

$$\mathcal{S}(n, v; a, b; \lambda) = \frac{1}{v!} \sum_{j=0}^v (-1)^j \binom{v}{j} \lambda^{v-j} (j \ln a + (v-j) \ln b)^n \quad (2.2)$$

and

$$\mathcal{S}(n, v; a, b; \lambda) = \frac{1}{v!} \sum_{j=0}^v (-1)^{v-j} \binom{v}{j} \lambda^j (j \ln b + (v-j) \ln a)^n. \quad (2.3)$$

Proof. By using (2.1) and the binomial theorem, we can easily arrive at the desired results. \square

By using the formula (2.2), we can compute some values of the numbers $\mathcal{S}(n, v; a, b; \lambda)$ as follows:

$$\begin{aligned} \mathcal{S}(0, 0; a, b; \lambda) &= 1, \\ \mathcal{S}(0, 0; a, b; \lambda) &= 1, \\ \mathcal{S}(1, 0; a, b; \lambda) &= 0, \\ \mathcal{S}(1, 1; a, b; \lambda) &= \ln \left(\frac{b^\lambda}{a} \right), \\ \mathcal{S}(2, 0; a, b; \lambda) &= 0, \\ \mathcal{S}(2, 1; a, b; \lambda) &= \lambda (\ln b)^2 - (\ln a)^2, \\ \mathcal{S}(2, 2; a, b; \lambda) &= \frac{\lambda^2}{2} (\ln b^2)^2 - \lambda \ln(ab) + (\ln a^2)^2, \\ \mathcal{S}(3, 0; a, b; \lambda) &= 0, \\ \mathcal{S}(3, 1; a, b; \lambda) &= \lambda (\ln b)^3 - (\ln a)^3, \\ \mathcal{S}(0, v; a, b; \lambda) &= \frac{(\lambda - 1)^v}{v!}, \\ \mathcal{S}(n, 0; a, b; \lambda) &= \delta_{n,0} \end{aligned}$$

and

$$\mathcal{S}(n, 1; a, b; \lambda) = \lambda (\ln b)^n - (\ln a)^n.$$

Remark 2. By setting $a = 1$ and $b = e$ in the assertions (2.2) of Theorem 1, we have the following result:

$$S(n, v; \lambda) = \frac{1}{v!} \sum_{j=0}^v \binom{v}{j} \lambda^{v-j} (-1)^j (v-j)^n.$$

The above relation has been studied by Srivastava [46] and Luo [29]. By setting $\lambda = 1$ in the above equation, we have the following result:

$$S(n, v; \lambda) = \frac{1}{v!} \sum_{j=0}^v \binom{v}{j} (-1)^j (v-j)^n$$

cf. ([1], [6], [8], [9], [13], [19], [29], [43], [44], [46], [48]).

By differentiating both sides of equation (2.1) with respect to the variable t , we obtain the following differential equations:

$$\frac{\partial}{\partial t} f_{S,v}(t; a, b; \lambda) = (\lambda(\ln b)b^t - (\ln a)a^t) f_{S,v-1}(t; a, b; \lambda)$$

or

$$\frac{\partial}{\partial t} f_{S,v}(t; a, b; \lambda) = v \ln(b) f_{S,v}(t; a, b; \lambda) + \ln\left(\frac{b}{a}\right) a^t f_{S,v-1}(t; a, b; \lambda). \quad (2.4)$$

By using equations (2.1) and (2.4), we obtain recurrence relations for the generalized λ -Stirling type numbers of the second kind by the following theorem:

Theorem 2. *Let $n, v \in \mathbb{N}$.*

$$\mathcal{S}(n, v; a, b; \lambda) = \sum_{j=0}^{n-1} \binom{n-1}{j} \mathcal{S}(j, v-1; a, b; \lambda) \left(\lambda(\ln(b))^{n-j} - (\ln(a))^{n-j} \right). \quad (2.5)$$

or

$$\begin{aligned} \mathcal{S}(n, v; a, b; \lambda) &= v \ln(b) \mathcal{S}(n-1, v; a, b; \lambda) \\ &\quad + \ln\left(\frac{b}{a}\right) \sum_{j=0}^{n-1} \binom{n-1}{j} \mathcal{S}(j, v-1; a, b; \lambda) (\ln(a))^{n-1-j}. \end{aligned}$$

Remark 3. *By setting $a = 1$ and $b = e$, Theorem 2 yields the corresponding results which are proven by Luo and Srivastava [29, Theorem 11]. Substituting $a = \lambda = 1$ and $b = e$ into Theorem 2, we obtain the following known results:*

$$S(n, v) = \sum_{j=0}^{n-1} \binom{n-1}{j} S(j, v-1),$$

and

$$S(n, v) = v S(n-1, v) + S(n-1, v-1),$$

cf. ([1], [10], [13], [29], [43], [44]).

The generalized λ -Stirling type numbers of the second kind can also be defined by equation (2.6):

Theorem 3. *Let $k \in \mathbb{N}_0$ and $\lambda \in \mathbb{C}$.*

$$\lambda^x (\ln b^x)^m = \sum_{l=0}^m \sum_{j=0}^{\infty} \binom{m}{l} \binom{x}{j} j! \mathcal{S}(l, j; a, b; \lambda) (\ln(a^{x-j}))^{m-l}. \quad (2.6)$$

Proof. By using (2.1), we get

$$(\lambda b^t)^x = \sum_{j=0}^{\infty} \binom{x}{j} j! \sum_{m=0}^{\infty} \mathcal{S}(m, j; a, b; \lambda) \frac{t^m}{m!} \sum_{n=0}^{\infty} (\ln a^{x-j})^n \frac{t^n}{n!}.$$

From the above equation, we obtain

$$\lambda^x \sum_{m=0}^{\infty} (\ln b)^m \frac{t^m}{m!} = \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} \binom{x}{j} j! \mathcal{S}(m, j; a, b; \lambda) \frac{t^m}{m!} \sum_{n=0}^{\infty} (\ln a^{x-j})^n \frac{t^n}{n!}.$$

Therefore

$$\lambda^x \sum_{m=0}^{\infty} (\ln b)^m \frac{t^m}{m!} = \sum_{m=0}^{\infty} \left(\sum_{l=0}^m \sum_{j=0}^{\infty} \binom{m}{l} \binom{x}{j} j! \mathcal{S}(l, j; a, b; \lambda) (\ln a^{(x-j)})^{m-l} \right) \frac{t^m}{m!}.$$

Comparing the coefficients of $\frac{t^m}{m!}$ on both sides of the above equation, we arrive at the desired result. \square

Remark 4. For $a = 0$ and $b = e$, the formula (2.6) can easily be shown to be reduced to the following result which is given by Luo and Srivastava [29, Theorem 9]:

$$\lambda^x x^n = \sum_{l=0}^{\infty} \binom{x}{l} l! S(n, l; \lambda),$$

where $n \in \mathbb{N}_0$ and $\lambda \in \mathbb{C}$. For $\lambda = 1$, the above formula is reduced to

$$x^n = \sum_{v=0}^n \binom{x}{v} v! S(n, v)$$

cf. ([1], [10], [13], [19], [29]).

3. GENERALIZED ARRAY TYPE POLYNOMIALS

By using the same motivation with the λ -Stirling type numbers of the second kind, we also construct a novel generating function, related to nonnegative real parameters, of the *generalized array type polynomials*. We derive some elementary properties including recurrence relations of these polynomials. The following definition provides a natural generalization and unification of the array polynomials:

Definition 2. Let $a, b \in \mathbb{R}^+$ ($a \neq b$), $x \in \mathbb{R}$, $\lambda \in \mathbb{C}$ and $v \in \mathbb{N}_0$. The generalized array type polynomials $\mathcal{S}_v^n(x; a, b; \lambda)$ can be defined by

$$\mathcal{S}_v^n(x; a, b; \lambda) = \frac{1}{v!} \sum_{j=0}^v (-1)^{v-j} \binom{v}{j} \lambda^j (\ln(a^{v-j} b^{x+j}))^n. \quad (3.1)$$

By using the formula (3.1), we can compute some values of the polynomials $\mathcal{S}_v^n(x; a, b; \lambda)$ as follows:

$$\begin{aligned} \mathcal{S}_0^n(x; a, b; \lambda) &= (\ln(b^x))^n, \\ \mathcal{S}_v^0(x; a, b; \lambda) &= \frac{(1 - \lambda)^v}{v!} \end{aligned}$$

and

$$\mathcal{S}_1^1(x; a, b; \lambda) = -\ln(ab^x) + \lambda \ln(b^{x+1}).$$

Remark 5. The polynomials $\mathcal{S}_v^n(x; a, b; \lambda)$ may be also called generalized λ -array type polynomials. By substituting $x = 0$ into (3.1), we arrive at (2.3):

$$\mathcal{S}_v^n(0; a, b; \lambda) = \mathcal{S}(n, v; a, b; \lambda).$$

Setting $a = \lambda = 1$ and $b = e$ in (3.1), we have

$$S_v^n(x) = \frac{1}{v!} \sum_{j=0}^v (-1)^{v-j} \binom{v}{j} (x+j)^n,$$

a result due to Chang and Ha [12, Eq-(3.1)], Simsek [43]. It is easy to see that

$$\begin{aligned} S_0^0(x) &= S_n^n(x) = 1, \\ S_0^n(x) &= x^n \end{aligned}$$

and for $v > n$,

$$S_v^n(x) = 0$$

cf. [12, Eq-(3.1)].

Generating functions for the polynomial $\mathcal{S}_v^n(x; a, b, c; \lambda)$ can be defined as follows:

Definition 3. Let $a, b \in \mathbb{R}^+$ ($a \neq b$), $\lambda \in \mathbb{C}$ and $v \in \mathbb{N}_0$. The generalized array type polynomials $\mathcal{S}_v^n(x; a, b; \lambda)$ are defined by means of the following generating function:

$$g_v(x, t; a, b; \lambda) = \sum_{n=0}^{\infty} \mathcal{S}_v^n(x; a, b; \lambda) \frac{t^n}{n!}. \quad (3.2)$$

Theorem 4. Let $a, b \in \mathbb{R}^+$, ($a \neq b$), $\lambda \in \mathbb{C}$ and $v \in \mathbb{N}_0$.

$$g_v(x, t; a, b; \lambda) = \frac{1}{v!} (\lambda b^t - a^t)^v b^{xt}. \quad (3.3)$$

Proof. By substituting (3.1) into the right hand side of (3.2), we obtain

$$\sum_{n=0}^{\infty} \mathcal{S}_v^n(x; a, b; \lambda) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left(\frac{1}{v!} \sum_{j=0}^v (-1)^{v-j} \binom{v}{j} \lambda^j (\ln(a^{v-j} b^{x+j}))^n \right) \frac{t^n}{n!}.$$

Therefore

$$\sum_{n=0}^{\infty} \mathcal{S}_v^n(x; a, b; \lambda) \frac{t^n}{n!} = \frac{1}{v!} \sum_{j=0}^v (-1)^{v-j} \binom{v}{j} \lambda^j \sum_{n=0}^{\infty} (\ln(a^{v-j} b^{x+j}))^n \frac{t^n}{n!}.$$

The right hand side of the above equation is the Taylor series for $e^{(\ln(a^{v-j} b^{x+j}))t}$, thus we get

$$\sum_{n=0}^{\infty} \mathcal{S}_v^n(x; a, b; \lambda) \frac{t^n}{n!} = \left(\frac{1}{v!} \sum_{j=0}^v (-1)^{v-j} \binom{v}{j} \lambda^j a^{(v-j)t} b^{jx} \right) b^{xt}.$$

By using (2.1) and binomial theorem in the above equation, we arrive at the desired result. \square

Remark 6. If we set $\lambda = 1$ in (3.3), we arrive a new special case of the array polynomials given by

$$f_{S,v}(t; a, b) b^{tx} = \sum_{n=0}^{\infty} \mathcal{S}_v^n(x; a, b) \frac{t^n}{n!}.$$

In the special case when

$$a = \lambda = 1 \text{ and } b = e,$$

the generalized array polynomials $\mathcal{S}_v^n(x; a, b; \lambda)$ defined by (3.3) would lead us at once to the classical array polynomials $\mathcal{S}_v^n(x)$, which are defined by means of the following generating function:

$$\frac{(e^t - 1)^v}{v!} e^{tx} = \sum_{n=0}^{\infty} \mathcal{S}_v^n(x) \frac{t^n}{n!},$$

which yields to the generating function for the array polynomials $\mathcal{S}_v^n(x)$ studied by Chang and Ha [12] see also cf. ([6], [43]).

The polynomials $\mathcal{S}_v^n(x; a, b; \lambda)$ defined by (3.3) have many interesting properties which we give in this section.

We set

$$g_v(x, t; a, b; \lambda) = b^{xt} f_{\mathcal{S}, v}(t; a, b; \lambda). \quad (3.4)$$

Theorem 5. *The following formula holds true:*

$$\mathcal{S}_v^n(x; a, b; \lambda) = \sum_{j=0}^n \binom{n}{j} \mathcal{S}(j, v; a, b; \lambda) (\ln b^x)^{n-j}. \quad (3.5)$$

Proof. By using (3.4), we obtain

$$\sum_{n=0}^{\infty} \mathcal{S}_v^n(x; a, b; \lambda) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \mathcal{S}(n, v; a, b; \lambda) \frac{t^n}{n!} \sum_{n=0}^{\infty} (\ln b^x)^n \frac{t^n}{n!}.$$

From the above equation, we get

$$\sum_{n=0}^{\infty} \mathcal{S}_v^n(x; a, b; \lambda) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left(\sum_{j=0}^n \binom{n}{j} \mathcal{S}(j, v; a, b; \lambda) (\ln b^x)^{n-j} \right) \frac{t^n}{n!}.$$

Comparing the coefficients of t^n on both sides of the above equation, we arrive at the desired result. \square

Remark 7. *In the special case when $a = \lambda = 1$ and $b = e$, equation (3.5) is reduced to*

$$\mathcal{S}_v^n(x) = \sum_{j=0}^n \binom{n}{j} x^{n-j} \mathcal{S}(j, v)$$

cf. [43, Theorem 2].

By differentiating j times both sides of (3.3) with respect to the variable x , we obtain the following differential equation:

$$\frac{\partial^j}{\partial x^j} g_v(x, t; a, b; \lambda) = t^j (\ln b)^j g_v(x, t; a, b; \lambda).$$

From this equation, we arrive at higher order derivative of the array type polynomials by the following theorem:

Theorem 6. *Let $n, j \in \mathbb{N}$ with $j \leq n$. Then we have*

$$\frac{\partial^j}{\partial x^j} \mathcal{S}_v^n(x; a, b; \lambda) = \{n\}_j (\ln(b))^j \mathcal{S}_v^{n-j}(x; a, b; \lambda).$$

Remark 8. By setting $a = \lambda = j = 1$ and $b = e$ in Theorem 6, we have

$$\frac{d}{dx} S_v^n(x) = n S_v^{n-1}(x)$$

cf. [43].

From (3.3), we get the following functional equation:

$$g_{v_1}(x_1, t; a, b; \lambda) g_{v_2}(x_2, t; a, b; \lambda) = \binom{v_1 + v_2}{v_1} g_{v_1+v_2}(x_1 + x_2, t; a, b; \lambda). \quad (3.6)$$

From this functional equation, we obtain the following identity:

Theorem 7.

$$\binom{v_1 + v_2}{v_1} \mathcal{S}_{v_1+v_2}^n(x_1 + x_2; a, b; \lambda) = \sum_{j=0}^n \binom{n}{j} \mathcal{S}_{v_1}^j(x_1; a, b; \lambda) \mathcal{S}_{v_2}^{n-j}(x_2; a, b; \lambda).$$

Proof. Combining (3.2) and (3.6), we get

$$\begin{aligned} & \sum_{n=0}^{\infty} \mathcal{S}_{v_1}^n(x_1; a, b; \lambda) \frac{t^n}{n!} \sum_{n=0}^{\infty} \mathcal{S}_{v_2}^n(x_2; a, b; \lambda) \frac{t^n}{n!} \\ &= \binom{v_1 + v_2}{v_1} \sum_{n=0}^{\infty} \mathcal{S}_{v_1+v_2}^n(x_1 + x_2; a, b; \lambda) \frac{t^n}{n!}. \end{aligned}$$

Therefore

$$\begin{aligned} & \sum_{n=0}^{\infty} \left(\sum_{j=0}^n \binom{n}{j} \mathcal{S}_{v_1}^j(x_1; a, b; \lambda) \mathcal{S}_{v_2}^{n-j}(x_2; a, b; \lambda) \right) \frac{t^n}{n!} \\ &= \binom{v_1 + v_2}{v_1} \sum_{n=0}^{\infty} \mathcal{S}_{v_1+v_2}^n(x_1 + x_2; a, b; \lambda) \frac{t^n}{n!}. \end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result. \square

4. GENERALIZED EULERIAN TYPE NUMBERS AND POLYNOMIALS

In this section, we provide generating functions, related to nonnegative real parameters, for the generalized Eulerian type polynomials and numbers, that is, the so called *generalized Apostol type Frobenius Euler polynomials and numbers*. We derive fundamental properties, recurrence relations and many new identities for these polynomials and numbers based on the generating functions, functional equations and differential equations.

These polynomials and numbers have many applications in many branches of Mathematics.

The following definition gives us a natural generalization of the Eulerian polynomials:

Definition 4. Let $a, b \in \mathbb{R}^+$ ($a \neq b$), $x \in \mathbb{R}$, $\lambda \in \mathbb{C}$ and $u \in \mathbb{C} \setminus \{1\}$. The generalized Eulerian type polynomials $\mathcal{H}_n(x; u; a, b, c; \lambda)$ are defined by means of the following generating function:

$$F_\lambda(t, x; u, a, b, c) = \frac{(a^t - u)c^{xt}}{\lambda b^t - u} = \sum_{n=0}^{\infty} \mathcal{H}_n(x; u; a, b, c; \lambda) \frac{t^n}{n!}. \quad (4.1)$$

By substituting $x = 0$ into (4.1), we obtain

$$\mathcal{H}_n(0; u; a, b, c; \lambda) = \mathcal{H}_n(u; a, b, c; \lambda),$$

where $\mathcal{H}_n(u; a, b, c; \lambda)$ denotes generalized Eulerian type numbers.

Remark 9. Substituting $a = 1$ into (4.1), we have

$$\frac{(1-u)c^{xt}}{\lambda b^t - u} = \sum_{n=0}^{\infty} \mathcal{H}_n(x; u; 1, b, c; \lambda) \frac{t^n}{n!}$$

a result due to Kurt and Simsek [25]. In their special case when $\lambda = 1$ and $b = c = e$, the generalized Eulerian type polynomials $\mathcal{H}_n(x; u; 1, b, c; \lambda)$ are reduced to the Eulerian polynomials or Frobenius Euler polynomials which are defined by means of the following generating function:

$$\frac{(1-u)e^{xt}}{e^t - u} = \sum_{n=0}^{\infty} H_n(x; u) \frac{t^n}{n!}, \quad (4.2)$$

with, of course, $H_n(0; u) = H_n(u)$ denotes the so-called Eulerian numbers cf. ([8], [7], [9], [10], [22], [45], [21], [39], [40], [47], [51]). Substituting $u = -1$, into (4.2), we have

$$H_n(x; -1) = E_n(x)$$

where $E_n(x)$ denotes Euler polynomials which are defined by means of the following generating function:

$$\frac{2e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!} \quad (4.3)$$

where $|t| < \pi$ cf. [1]-[53].

The following elementary properties of the generalized Eulerian type polynomials and numbers are derived from their generating functions in (4.1).

Theorem 8. (Recurrence relation for the generalized Eulerian type numbers): For $n = 0$, we have

$$\mathcal{H}_0(u; a, b; \lambda) = \begin{cases} \frac{1-u}{\lambda-u} & \text{if } a = 1, \\ \frac{u}{\lambda-u} & \text{if } a \neq 1. \end{cases}$$

For $n > 0$, following the usual convention of symbolically replacing $(\mathcal{H}(u; a, b; \lambda))^n$ by $\mathcal{H}_n(u; a, b; \lambda)$, we have

$$\lambda (\ln b + \mathcal{H}(u; a, b; \lambda))^n - u \mathcal{H}_n(u; a, b; \lambda) = (\ln a)^n.$$

Proof. By using (4.1), we obtain

$$\sum_{n=0}^{\infty} (\ln a)^n \frac{t^n}{n!} - u = \sum_{n=0}^{\infty} (\lambda (\ln b + \mathcal{H}(u; a, b; \lambda))^n - u \mathcal{H}_n(u; a, b; \lambda)) \frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result. \square

By differentiating both sides of equation (4.1) with respect to the variable x , we obtain the following higher order differential equation:

$$\frac{\partial^j}{\partial x^j} F_{\lambda}(t, x; u, a, b, c) = (\ln(c))^j F_{\lambda}(t, x; u, a, b, c). \quad (4.4)$$

From this equation, we arrive at higher order derivative of the generalized Eulerian type polynomials by the following theorem:

Theorem 9. *Let $n, j \in \mathbb{N}$ with $j \leq n$. Then we have*

$$\frac{\partial^j}{\partial x^j} \mathcal{H}_n(x; u; a, b, c; \lambda) = \{n\}_j (\ln(c))^j \mathcal{H}_{n-j}(x; u; a, b, c; \lambda).$$

Proof. Combining (4.1) and (4.4), we have

$$\sum_{n=0}^{\infty} \frac{\partial^j}{\partial x^j} \mathcal{H}_n(x; u; a, b, c; \lambda) \frac{t^n}{n!} = (\ln c)^j \sum_{n=0}^{\infty} \mathcal{H}_n(x; u; a, b, c; \lambda) \frac{t^{n+j}}{n!}.$$

From the above equation, we get

$$\sum_{n=0}^{\infty} \frac{\partial^j}{\partial x^j} \mathcal{H}_n(x; u; a, b, c; \lambda) \frac{t^n}{n!} = (\ln c)^j \sum_{n=0}^{\infty} \{n\}_j \mathcal{H}_{n-j}(x; u; a, b, c; \lambda) \frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result. \square

Remark 10. *Setting $j = 1$ in Theorem 9, we have*

$$\frac{\partial}{\partial x} \mathcal{H}_n(x; u; a, b, c; \lambda) = n \mathcal{H}_{n-1}(x; u; a, b, c; \lambda) \ln(c).$$

In their special case when

$$a = \lambda = 1 \text{ and } b = c = e,$$

Theorem 9 is reduced to the following well known result:

$$\frac{\partial^j}{\partial x^j} H_n(x; u) = \frac{n!}{(n-j)!} H_{n-j}(x; u)$$

cf. [8, Eq-(3.5)]. Substituting $j = 1$ into the above equation, we have

$$\frac{\partial}{\partial x} H_n(x; u) = n H_{n-1}(x; u)$$

cf. ([8, Eq-(3.5)], [25]).

Theorem 10. *The following explicit representation formula holds true:*

$$\begin{aligned} & (x \ln c + \ln a)^n - u x^n (\ln c)^n \\ &= \lambda (x \ln c + \ln b + \mathcal{H}(u; a, b; \lambda))^n - u (x \ln c + \mathcal{H}(u; a, b; \lambda))^n. \end{aligned}$$

Proof. By using (4.1) and the *umbral calculus convention*, we obtain

$$\frac{a^t - u}{\lambda b^t - u} = e^{H(u; a, b; \lambda)t}.$$

From the above equation, we get

$$\begin{aligned} & \sum_{n=0}^{\infty} ((\ln a + x \ln c)^n - u (x \ln c)^n) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} (\lambda (\mathcal{H}(u; a, b; \lambda) + \ln b + x \ln c)^n - u (\mathcal{H}_n(u; a, b; \lambda) + x \ln c)^n) \frac{t^n}{n!}. \end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result. \square

Remark 11. *By substituting $a = \lambda = 1$ and $b = c = e$ into Theorem 10, we have*

$$(1 - u) x^n = H_n(x + 1; u) - u H_n(x; u) \quad (4.5)$$

cf. ([8, Eq-(3.3)], [51]). *By setting $u = -1$ in the above equation, we have*

$$2x^n = E_n(x + 1) + E_n(x)$$

*a result due to Shiratani [38]. By using (4.5), Carlitz [8] studied on the **Mirimonoff polynomial** $f_n(0, m)$ which is defined by*

$$\begin{aligned} f_n(x, m) &= \sum_{j=0}^{m-1} (x + j)^n u^{m-j-1} \\ &= \frac{H_n(x + m; u) - u^m H_n(x; u)}{1 - u}. \end{aligned}$$

By applying Theorem 10, one may generalize the Mirimonoff polynomial.

Theorem 11. *The following explicit representation formula holds true:*

$$\mathcal{H}_n(x; u; a, b, c; \lambda) = \sum_{j=0}^n \binom{n}{j} (x \ln c)^{n-j} \mathcal{H}_j(u; a, b, c; \lambda). \quad (4.6)$$

Proof. By using (4.1), we get

$$\sum_{n=0}^{\infty} \mathcal{H}_n(u; a, b, c; \lambda) \frac{t^n}{n!} \sum_{n=0}^{\infty} (\ln c)^n \frac{t^n}{n!} = \sum_{n=0}^{\infty} \mathcal{H}_n(x; u; a, b, c; \lambda) \frac{t^n}{n!}.$$

From the above equation, we obtain

$$\sum_{n=0}^{\infty} \left(\sum_{j=0}^n \binom{n}{j} (x \ln c)^{n-j} \mathcal{H}_j(u; a, b, c; \lambda) \right) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \mathcal{H}_n(x; u; a, b, c; \lambda) \frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result. \square

Remark 12. Substituting $a = \lambda = 1$ and $b = c = e$ into (4.6), we have

$$H_n(x; u) = \sum_{j=0}^n \binom{n}{j} x^{n-j} H_j(u)$$

cf. ([8], [7], [9], [10], [22], [45], [21], [25], [39], [40], [47], [51]).

Remark 13. From (4.6), we easily get

$$\mathcal{H}_n(x; u; a, b, c; \lambda) = (\mathcal{H}(u; a, b, c; \lambda) + x \ln c)^n,$$

where after expansion of the right member, $\mathcal{H}^n(u; a, b, c; \lambda)$ is replaced by $\mathcal{H}_n(u; a, b, c; \lambda)$, we use this convention frequently throughout of this paper.

Theorem 12.

$$\mathcal{H}_n(x + y; u; a, b, c; \lambda) = \sum_{j=0}^n \binom{n}{j} (y \ln c)^{n-j} \mathcal{H}_j(x; u; a, b, c; \lambda). \quad (4.7)$$

Proof. By using (4.1), we have

$$\sum_{n=0}^{\infty} \mathcal{H}_n(x + y; u; a, b, c; \lambda) \frac{t^n}{n!} = \sum_{n=0}^{\infty} (y \ln c)^n \frac{t^n}{n!} \cdot \sum_{n=0}^{\infty} \mathcal{H}_n(x; u; a, b, c; \lambda) \frac{t^n}{n!}.$$

Therefore

$$\sum_{n=0}^{\infty} \mathcal{H}_n(x + y; u; a, b, c; \lambda) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \sum_{j=0}^n \binom{n}{j} (y \ln c)^{n-j} \mathcal{H}_j(x; u; a, b, c; \lambda) \frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result. \square

Remark 14. In the special case when $a = \lambda = 1$ and $b = c = e$, equation (4.7) is reduced to the following result:

$$H_n(x + y) = \sum_{j=0}^n \binom{n}{j} y^{n-j} H_j(x, u)$$

cf. [8, Eq.(3.6)]. Substituting $u = -1$ into the above equation, we get the following well-known result:

$$E_n(x + y) = \sum_{j=0}^n \binom{n}{j} y^{n-j} E_j(x). \quad (4.8)$$

By using (4.1), we define the following functional equation:

$$F_{\lambda^2}(t, x; u^2, a^2, b^2, c) c^{yt} = F_{\lambda}(t, x; u, a, b, c) F_{\lambda}(t, y; -u, a, b, c). \quad (4.9)$$

Theorem 13.

$$\mathcal{H}_n(x + y; u^2; a, b, c; \lambda^2) = (\mathcal{H}(x; u; a, b, c; \lambda) + \mathcal{H}(y; -u; a, b, c; \lambda))^n. \quad (4.10)$$

Proof. Combining (4.9) and (4.7), we easily arrive at the desired result. \square

Remark 15. In the special case when $a = \lambda = 1$ and $b = c = e$, equation (4.10) is reduced to the following result:

$$H_n(x + y; u^2) = \sum_{j=0}^n \binom{n}{j} H_j(x; u) H_{n-j}(y; -u)$$

cf. [8, Eq-(3.17)].

Theorem 14.

$$(-1)^n \mathcal{H}_n(1 - x; u^{-1}; a, b, c; \lambda^{-1}) = \lambda \sum_{j=0}^n \binom{n}{j} \left(\ln \left(\frac{b}{a} \right) \right)^{n-j} \mathcal{H}_j(x - 1, u; a, b, c; \lambda).$$

Proof. By using (4.1), we obtain

$$\frac{(a^{-t} - u^{-1}) c^{-(1-x)t}}{\lambda^{-1} b^{-t} - u^{-1}} = \lambda \left(\frac{b}{a} \right)^t \sum_{n=0}^{\infty} \mathcal{H}_n(x - 1; u; a, b, c; \lambda) \frac{t^n}{n!}.$$

From the above equation, we get

$$\begin{aligned} & \sum_{n=0}^{\infty} \mathcal{H}_n(1 - x; u^{-1}; a, b, c; \lambda^{-1}) \frac{(-1)^n t^n}{n!} \\ &= \lambda \left(\sum_{n=0}^{\infty} \mathcal{H}_n(x - 1; u; a, b, c; \lambda) \frac{t^n}{n!} \right) \left(\sum_{n=0}^{\infty} \left(\ln \left(\frac{b}{a} \right) \right)^n \frac{t^n}{n!} \right). \end{aligned}$$

Therefore

$$\begin{aligned} & \sum_{n=0}^{\infty} (-1)^n \mathcal{H}_n(1 - x; u^{-1}; a, b, c; \lambda^{-1}) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\lambda \sum_{j=0}^n \binom{n}{j} \left(\ln \left(\frac{b}{a} \right) \right)^{n-j} \mathcal{H}_j(x - 1, u; a, b, c; \lambda) \right) \frac{t^n}{n!}. \end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result. \square

Remark 16. In their special case when $a = \lambda = 1$ and $b = c = e$, Theorem 14 is reduced to the following result:

$$(-1)^n H_n(1 - x; u^{-1}) = H_n(x - 1, u)$$

cf. [8, Eq-(3.7)]. Substituting $u = -1$ into the above equation, we get the following well-known result:

$$(-1)^n E_n(1 - x) = E_n(x)$$

cf. ([8, Eq-(3.7)], [15], [36], [38], [46]).

Theorem 15.

$$\mathcal{H}_n \left(\frac{x+y}{2}; u^2; a, b, c; \lambda^2 \right) = \sum_{j=0}^n \binom{n}{j} \frac{\mathcal{H}_j(x; u; a, b, c; \lambda) \mathcal{H}_{n-j}(y; -u; a, b, c; \lambda)}{2^n}.$$

Proof. By using (4.1), we get

$$\begin{aligned} & \sum_{n=0}^{\infty} \mathcal{H}_n \left(\frac{x+y}{2}; u^2; a, b, c; \lambda^2 \right) \frac{2^n t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{j=0}^n \binom{n}{j} \mathcal{H}_j(x; u; a, b, c; \lambda) \mathcal{H}_{n-j}(y; -u; a, b, c; \lambda) \right) \frac{t^n}{n!}. \end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result. \square

Remark 17. When $a = \lambda = 1$ and $b = c = e$, Theorem 15 is reduced to the following result:

$$\mathcal{H}_n \left(\frac{x+y}{2}; u^2 \right) = 2^{-n} \sum_{j=0}^n \binom{n}{j} \mathcal{H}_j(x; u) \mathcal{H}_{n-j}(y; -u),$$

cf. [8, Eq-(3.17)].

4.1. Multiplication formulas for normalized polynomials. In this section, using generating functions, we derive *multiplication formulas* in terms of the normalized polynomials which are related to the generalized Eulerian type polynomials, the Bernoulli and the Euler polynomials.

Theorem 16. (Multiplication formula) Let $y \in \mathbb{N}$. Then we have

$$\begin{aligned} & \mathcal{H}_n(yx; u; a, b, b; \lambda) \\ &= y^n \sum_{k=0}^n \sum_{j=0}^{y-1} \binom{n}{k} \frac{\lambda^j (\ln a)^{n-k}}{u^{j+1-y} - u^{j+1}} \mathcal{H}_k \left(x + \frac{j}{y}; u^y; a, b, b; \lambda^y \right) \\ & \quad \times \left(H_{n-k} \left(\frac{1}{y}; u^y \right) - u H_{n-k}(u^y) \right), \end{aligned} \tag{4.11}$$

where $H_n(x; u)$ and $H_n(u)$ denote the Eulerian polynomials and numbers, respectively.

Proof. Substituting $c = b$ into (4.1), we have

$$\sum_{n=0}^{\infty} \mathcal{H}_n(x; u; a, b, b; \lambda) \frac{t^n}{n!} = \frac{(a^t - u) b^{xt}}{\lambda b^t - u} = \left(\frac{a^t - u}{-u} \right) \frac{(a^t - u) b^{xt}}{1 - \frac{\lambda b^t}{u}}. \tag{4.12}$$

By using the following finite geometric series

$$\sum_{j=0}^{y-1} \left(\frac{\lambda b^t}{u} \right)^j = \frac{1 - \left(\frac{\lambda b^t}{u} \right)^y}{1 - \frac{\lambda b^t}{u}},$$

on the right-hand side of (4.12), we obtain

$$\sum_{n=0}^{\infty} \mathcal{H}_n(x; u; a, b, b; \lambda) \frac{t^n}{n!} = \frac{(a^t - u) b^{xt}}{-u \left(1 - \left(\frac{\lambda b^t}{u}\right)^y\right)} \sum_{j=0}^{y-1} \left(\frac{\lambda b^t}{u}\right)^j.$$

From this equation, we get

$$\sum_{n=0}^{\infty} \mathcal{H}_n(x; u; a, b, b; \lambda) \frac{t^n}{n!} = \frac{(a^t - u)}{(a^{yt} - u^y)} \sum_{j=0}^{y-1} \frac{\lambda^j}{u^{j+1-y}} \frac{(a^{yt} - u^y) b^{yt \left(\frac{x+j}{y}\right)}}{(\lambda b^{yt} - u^y)}.$$

Now by making use of the generating functions (4.1) and (4.2) on the right-hand side of the above equation, we obtain

$$\begin{aligned} & \sum_{n=0}^{\infty} \mathcal{H}_n(x; u; a, b, b; \lambda) \frac{t^n}{n!} \\ &= \frac{1}{1 - u^y} \sum_{j=0}^{y-1} \frac{\lambda^j}{u^{j+1-y}} \left(\sum_{n=0}^{\infty} \mathcal{H}_n\left(\frac{x+j}{y}; u^y; a, b, b; \lambda^y\right) \frac{y^n t^n}{n!} \right) \\ & \quad \times \left(\sum_{n=0}^{\infty} \left(H_n\left(\frac{1}{y}; u^y\right) - u H_n(u^y) \right) \frac{(y \ln a)^n t^n}{n!} \right). \end{aligned}$$

Therefore

$$\begin{aligned} & \sum_{n=0}^{\infty} \mathcal{H}_n(x; u; a, b, b; \lambda) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^n \sum_{j=0}^{y-1} \binom{n}{k} \frac{y^n \lambda^j (\ln a)^{n-k}}{u^{j+1-y} - u^{j+1}} \mathcal{H}_k\left(\frac{x+j}{y}; u^y; a, b, b; \lambda^y\right) \\ & \quad \times \left(H_{n-k}\left(\frac{1}{y}; u^y\right) - u H_{n-k}(u^y) \right) \frac{t^n}{n!}. \end{aligned}$$

By equating the coefficients of $\frac{t^n}{n!}$ on both sides, we get

$$\begin{aligned} & \mathcal{H}_n(x; u; a, b, b; \lambda) \\ &= \sum_{k=0}^n \sum_{j=0}^{y-1} \binom{n}{k} \frac{y^n \lambda^j (\ln a)^{n-k}}{u^{j+1-y} - u^{j+1}} \mathcal{H}_k\left(\frac{x+j}{y}; u^y; a, b, b; \lambda^y\right) \\ & \quad \times \left(H_{n-k}\left(\frac{1}{y}; u^y\right) - u H_{n-k}(u^y) \right). \end{aligned}$$

Finally, by replacing x by yx on both sides of the above equation, we arrive at the desired result. \square

Remark 18. By substituting $a = 1$ into Theorem 16, for $n = k$, we obtain

$$\mathcal{H}_n(yx; u; 1, b, b; \lambda) = y^n u^{y-1} \frac{1-u}{1-u^y} \sum_{j=0}^{y-1} \frac{\lambda^j}{u^j} \mathcal{H}_n\left(x + \frac{j}{y}; u^y; 1, b, b; \lambda^y\right). \quad (4.13)$$

By substituting $b = e$ and $\lambda = 1$ into the above equation, we arrive at the multiplication formula for the Eulerian polynomials

$$H_n(yx; u) = y^n u^{y-1} \frac{(1-u)}{1-uy} \sum_{j=0}^{y-1} \frac{1}{u^j} H_n \left(x + \frac{j}{y}; u^y \right), \quad (4.14)$$

cf. ([9], [8, Eq-(3.12)]). If $u = -1$, then the above equation reduces to the well known multiplication formula for the Euler polynomials: for y is an odd positive integer, we have

$$E_n(yx) = y^n \sum_{j=0}^{y-1} (-1)^j E_n \left(x + \frac{j}{y} \right), \quad (4.15)$$

where $E_n(x)$ denotes the Euler polynomials in the usual notation. If y is an even positive integer, we have

$$E_n(yx) = \frac{2y^{n-1}}{n} \sum_{j=0}^{y-1} (-1)^j B_n \left(x + \frac{j}{y} \right), \quad (4.16)$$

where $B_n(x)$ and $E_n(x)$ denote the Bernoulli polynomials and Euler polynomials, respectively, cf. ([7], [50]).

To prove the multiplication formula of the generalized Apostol Bernoulli polynomials, we need the following generating function which is defined by Srivastava et al. [48, pp. 254, Eq. (20)]:

Definition 5. Let $a, b, c \in \mathbb{R}^+$ with $a \neq b$, $x \in \mathbb{R}$ and $n \in \mathbb{N}_0$. Then the generalized Bernoulli polynomials $\mathfrak{B}_n^{(\alpha)}(x; \lambda; a, b, c)$ of order $\alpha \in \mathbb{C}$ are defined by means of the following generating functions:

$$f_B(x, a, b, c; \lambda; \alpha) = \left(\frac{t}{\lambda b^t - a^t} \right)^\alpha c^{xt} = \sum_{n=0}^{\infty} \mathfrak{B}_n^{(\alpha)}(x; \lambda; a, b, c) \frac{t^n}{n!}, \quad (4.17)$$

where

$$\left| t \ln \left(\frac{a}{b} \right) + \ln \lambda \right| < 2\pi$$

and

$$1^\alpha = 1.$$

Observe that if we set $\lambda = 1$ in (4.17), we have

$$\left(\frac{t}{b^t - a^t} \right)^\alpha c^{xt} = \sum_{n=0}^{\infty} \mathfrak{B}_n^{(\alpha)}(x; a, b, c) \frac{t^n}{n!}. \quad (4.18)$$

If we set $x = 0$ in (4.18), we obtain

$$\left(\frac{t}{b^t - a^t} \right)^\alpha = \sum_{n=0}^{\infty} \mathfrak{B}_n^{(\alpha)}(a, b) \frac{t^n}{n!}, \quad (4.19)$$

with of course, $\mathfrak{B}_n^{(\alpha)}(x; a, b, c) = \mathfrak{B}_n^{(\alpha)}(a, b)$, cf. ([30]-[31], [21], [45], [26], [32], [34], [35], [47], [49], [46], [48]). If we set $\alpha = 1$ in (4.19) and (4.18), we have

$$\frac{t}{b^t - a^t} = \sum_{n=0}^{\infty} \mathfrak{B}_n(a, b) \frac{t^n}{n!} \quad (4.20)$$

and

$$\left(\frac{t}{b^t - a^t} \right) c^{xt} = \sum_{n=0}^{\infty} \mathfrak{B}_n(x; a, b, c) \frac{t^n}{n!}, \quad (4.21)$$

which have been studied by Luo et al. [30]-[31]. Moreover, by substituting $a = 1$ and $b = c = e$ into (4.17), then we arrive at the Apostol-Bernoulli polynomials $\mathcal{B}_n(x; \lambda)$, which are defined by means of the following generating function

$$\left(\frac{t}{\lambda e^t - 1} \right) e^{xt} = \sum_{n=0}^{\infty} \mathcal{B}_n(x; \lambda) \frac{t^n}{n!},$$

These polynomials $\mathcal{B}_n(x; \lambda)$ have been introduced and investigated by many Mathematicians cf. ([3], [23], [18], [21], [25], [28], [35], [41], [47]). When $a = \lambda = 1$ and $b = c = e$ into (4.20) and (4.21), $\mathfrak{B}_n(a, b)$ and $\mathfrak{B}_n(x; a, b, c)$ reduce to the classical Bernoulli numbers and the classical Bernoulli polynomials, respectively, cf. [1]-[53].

Remark 19. *The constraints on $|t|$, which we have used in Definition 5 and (4.3), are meant to ensure that the generating function in (4.18) and (4.3) are analytic throughout the prescribed open disks in complex t -plane (centred at the origin $t = 0$) in order to have the corresponding convergent Taylor-Maclaurin series expansion (about the origin $t = 0$) occurring on the their right-hand side (with a positive radius of convergence) cf. [49].*

Theorem 17. *Let $y \in \mathbb{N}$. Then we have*

$$\mathfrak{B}_n(yx; \lambda; a, b, b) = \sum_{l=0}^n \sum_{j=0}^{y-1} \binom{n}{l} \lambda^j y^{l-1} ((y-1-j) \ln a)^{n-l} \mathfrak{B}_l \left(x + \frac{j}{y}; \lambda^y; a, b, b \right).$$

Proof. Substituting $c = b$ and $\alpha = 1$ into (4.17), we get

$$\sum_{n=0}^{\infty} \mathfrak{B}_n(x; \lambda; a, b, c) \frac{t^n}{n!} = \frac{1}{y} \sum_{j=0}^{y-1} \lambda^j \frac{yt}{\lambda^y b^{yt} - a^{yt}} b^{\left(\frac{x+j}{y}\right)yt} a^{t(y-j-1)}.$$

Therefore

$$\begin{aligned} & \sum_{n=0}^{\infty} \mathfrak{B}_n(x; \lambda; a, b, c) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \sum_{l=0}^n \sum_{j=0}^{y-1} \binom{n}{l} \lambda^j ((y-1-j) \ln a)^{n-l} y^{l-1} \mathfrak{B}_l \left(\frac{x+j}{y}; \lambda^y; a, b, b \right) \frac{t^n}{n!}. \end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we get

$$\mathfrak{B}_n^{(\alpha)}(x; \lambda; a, b, c) = \sum_{l=0}^n \sum_{j=0}^{y-1} \binom{n}{k} \lambda^j ((k-1-j) \ln a)^{n-l} y^{l-1} \mathfrak{B}_l \left(\frac{x+j}{k}; \lambda^y; a, b, b \right).$$

By replacing x by yx on both sides of the above equation, we arrive at the desired result. \square

Remark 20. Kurt and Simsek [26] proved multiplication formula for the generalized Bernoulli polynomials of order α . When $a = \lambda = 1$ and $b = c = e$ into Theorem 17, we have the multiplication formula for the Bernoulli polynomials given by

$$B_n(yx) = y^{n-1} \sum_{j=0}^{y-1} B_n \left(x + \frac{j}{y} \right), \quad (4.22)$$

cf. ([3], [7], [8], [14], [21], [31], [28], [27], [32], [29], [47], [48]).

If f is a *normalized* polynomial such that it satisfies the formula

$$f_n(yx) = y^{n-1} \sum_{j=0}^{y-1} f_n \left(x + \frac{j}{y} \right), \quad (4.23)$$

then f is the y th degree Bernoulli polynomial due to (4.22) cf. ([7], [53]). According to Nielsen [7], if a normalized polynomial satisfies (4.22) for a single value of $y > 1$, then it is identical with $B_m(x)$. Consequently, if a normalized polynomial satisfies (4.13) for a single value of $y > 1$, then it is identical with $\mathcal{H}_n(x; u; 1, b, b; \lambda)$. The formula (4.16) is different. Therefore, for y is an even positive integer, Carlitz [7, Eq-(1.4)] considered the following equation:

$$g_{n-1}(yx) = -\frac{2y^{n-1}}{n} \sum_{j=0}^{y-1} (-1)^j f_n \left(x + \frac{j}{y} \right),$$

where $g_{n-1}(x)$ and $f_n(x)$ denote the normalized polynomials of degree $n-1$ and n , respectively. More precisely, as Carlitz has pointed out [7, p. 184], if y is a fixed even integer ≥ 2 and $f_n(x)$ is an arbitrary normalized polynomial of degree n , then (4.16) determines $g_{n-1}(x)$ as a normalized polynomial of degree $n-1$. Thus, for a single value y , (4.16) does not suffice to determine the normalized polynomials $g_{n-1}(x)$ and $f_n(x)$.

Remark 21. According to (4.23), the set of normalized polynomials $\{f_n(x)\}$ is an Appell set, cf. [7].

We now modify (4.1) as follows:

$$\frac{(a^t - \xi) c^{xt}}{\lambda b^t - \xi} = \sum_{n=0}^{\infty} \mathcal{H}_n(x; \xi; a, b, c; \lambda) \frac{t^n}{n!} \quad (4.24)$$

where

$$\xi^r = 1, \quad \xi \neq 1.$$

The polynomial $\mathcal{H}_n(x; \xi; a, b, c; \lambda)$ is a normalized polynomial of degree m in x . The polynomial $H_n(x; \xi; 1, e, e; 1)$ may be called Eulerian polynomials with parameter ξ . In particular we note that

$$\mathcal{H}_n(x; -1; 1, e, e; 1) = E_n(x)$$

since for $a = \lambda = 1$, $b = c = e$, equation (4.24) reduces to the generating function for the Euler polynomials.

By means of equation (4.11), it is easy to verify the following multiplication formulas:

If y is an odd positive integer, then we have

$$\begin{aligned} \mathcal{H}_{n-1}(yx; \xi; a, b, b; \lambda) &= \frac{y^{n-1}}{n} \sum_{j=0}^{y-1} \left(\frac{\lambda}{\xi}\right)^j \mathfrak{B}_n\left(x + \frac{j}{y}; b; \lambda^y\right) \\ &\quad - \frac{1}{\xi n} \sum_{k=0}^n \sum_{j=0}^{y-1} \left(\frac{\lambda}{\xi}\right)^j y^{k-1} (\ln a)^{n-k} \mathfrak{B}_k\left(x + \frac{j}{y}; b; \lambda^y\right), \end{aligned} \quad (4.25)$$

where

$$\mathcal{H}_k\left(x + \frac{j}{y}; \xi^y; 1, b, b; \lambda^y\right) = \mathfrak{B}_n\left(x + \frac{j}{y}; b; \lambda^y\right).$$

If y is an even positive integer, then we have

$$\begin{aligned} \mathcal{H}_n(yx; \xi; a, b, b; \lambda) &= \frac{y^n}{2} \sum_{j=0}^{y-1} \left(\frac{\lambda}{\xi}\right)^j \mathfrak{E}_n\left(x + \frac{j}{y}; b; \lambda^y\right) \\ &\quad - \frac{1}{2\xi} \sum_{k=0}^n \sum_{j=0}^{y-1} \left(\frac{\lambda}{\xi}\right)^j y^k (\ln a)^{n-k} \mathfrak{E}_k\left(x + \frac{j}{y}; b; \lambda^y\right), \end{aligned} \quad (4.26)$$

where

$$\mathcal{H}_k\left(x + \frac{j}{y}; \xi^y; 1, b, b; \lambda^y\right) = \mathfrak{E}_n\left(x + \frac{j}{y}; b; \lambda^y\right),$$

where $\mathfrak{E}_n(x; a, b, c)$ denotes the generalized Euler polynomials, which are defined by means of the following generating function:

$$\left(\frac{t}{b^t - a^t}\right) c^{xt} = \sum_{n=0}^{\infty} \mathfrak{E}_n(x; a, b, c) \frac{t^n}{n!}$$

cf. ([30]-[31], [24], [26], [35], [47], [49], [46], [48]).

Remark 22. If we set $a = \lambda = 1$ and $b = e$, then (4.25) and (4.26) reduce to the following multiplication formulas, respectively:

$$H_{n-1}(yx; \xi) = \frac{y^{n-1}}{n} \left(1 - \frac{1}{\xi}\right) \sum_{j=0}^{y-1} \frac{1}{\xi^j} B_n\left(x + \frac{j}{y}\right)$$

cf. [7, Eq. (3.3)] and

$$H_n(yx; \xi) = \frac{y^n}{2} \left(1 - \frac{1}{\xi}\right) \sum_{j=0}^{y-1} \frac{1}{\xi^j} E_n\left(x + \frac{j}{y}\right).$$

Let $f_n(x)$ and $g_n(x)$ be normalized polynomials in the usual way. Carlitz [7, Eq. (3.4)] defined the following equation:

$$g_{n-1}(yx) = \frac{(1-\rho)y^{n-1}}{n} \sum_{j=0}^{y-1} \rho^j f_n\left(x + \frac{j}{y}\right),$$

where ρ is a fixed primitive r th root of unity, $r > 1$, $y \equiv 0 \pmod{r}$.

Remark 23. If we set $a = \lambda = 1$, $b = c = e$ and $\xi = -1$, then (4.25) and (4.26) reduce to (4.16) and (4.15).

Remark 24. Walum [53] defined multiplication formula for periodic functions as follows:

$$\vartheta(y)f(yx) = \sum_{j(y)} f\left(x + \frac{j}{y}\right), \quad (4.27)$$

where f is periodic with period 1 and $j(y)$ under the summation sign indicates that j runs through a complete system of residues mod y .

Formulas (4.23), (4.27) and other multiplication formulas related to periodic functions and normalized polynomials occur in Franel's formula, in the theory of the Dedekind sums and Hardy-Berndt sums, in the theory of the zeta functions and L -functions and in the theory of periodic bounded variation, cf. ([4], [5], [53]).

4.2. Generalized Eulerian type numbers and polynomials attached to Dirichlet character. In this section, we construct generating function, related to nonnegative real parameters, for the generalized Eulerian type numbers and polynomials attached to Dirichlet character. We also give some properties of these polynomials and numbers.

Definition 6. Let χ be the Dirichlet character of conductor $f \in \mathbb{N}$. Let $x \in \mathbb{R}$, $a, b \in \mathbb{R}^+$, ($a \neq b$), $\lambda \in \mathbb{C}$ and $u \in \mathbb{C} \setminus \{1\}$. The generalized Eulerian type polynomials $\mathcal{H}_{n,\chi}(x; u; a, b, c; \lambda)$ are defined by means of the following generating function:

$$\mathcal{F}_{\lambda,\chi}(t, x; u, a, b, c) = \sum_{j=0}^{f-1} \frac{(a^{ft} - u^f) \chi(j) u^{f-j-1} c^{\left(\frac{x+j}{f}\right)ft}}{\lambda^f b^{ft} - u^f} = \sum_{n=0}^{\infty} \mathcal{H}_{n,\chi}(x; u; a, b, c; \lambda) \frac{t^n}{n!} \quad (4.28)$$

with, of course

$$\mathcal{H}_{n,\chi}(0; u; a, b, c; \lambda) = \mathcal{H}_{n,\chi}(u; a, b, c; \lambda),$$

where $\mathcal{H}_{n,\chi}(u; a, b, c; \lambda)$ denotes generalized Eulerian type numbers.

Remark 25. In the special case when $a = \lambda = 1$ and $b = c = e$, the generalized Eulerian type polynomials $\mathcal{H}_{n,\chi}(x; u; a, b, c; \lambda)$ are reduced to the Frobenius Euler polynomials which are defined by means of the following generating function:

$$\sum_{j=0}^{f-1} \frac{(1 - u^f) \chi(j) u^{f-j-1} e^{\left(\frac{x+j}{f}\right)ft}}{e^{ft} - u^f} = \sum_{n=0}^{\infty} H_{n,\chi}(x; u) \frac{t^n}{n!},$$

cf. ([51], [21], [45], [39], [40], [47]). Substituting $u = -1$ into the above equation, we have generating function of the generalized Euler polynomials attached to Dirichlet character with

odd conductor:

$$2 \sum_{j=0}^{f-1} \frac{\chi(j)(-1)^j e^{\left(\frac{x+j}{f}\right)ft}}{e^{ft} + 1} = \sum_{n=0}^{\infty} E_{n,\chi}(x) \frac{t^n}{n!},$$

cf. ([51], [39], [40], [47]).

Combining (4.1) and (4.28), we obtain the following functional equation:

$$\mathcal{F}_{\lambda,\chi}(t, x; u, a, b, c;) = \sum_{j=0}^{f-1} \chi(j) u^{f-j-1} F_{\lambda^f}(ft, \frac{x+j}{f}; u^f, a, b, c).$$

By using the above functional equation we arrive at the following Theorem:

Theorem 18.

$$\mathcal{H}_{n,\chi}(x; u; a, b, c; \lambda) = f^n \sum_{j=0}^{f-1} \chi(j) u^{f-j-1} \mathcal{H}_n(\frac{x+j}{f}; u^f; a, b, c; \lambda^f).$$

Theorem 19.

$$\mathcal{H}_{n,\chi}(x; u; a, b, c; \lambda) = \sum_{j=0}^n \binom{n}{j} (x \ln c)^{n-j} \mathcal{H}_{j,\chi}(u; a, b, c; \lambda).$$

Proof. By using (4.28), we get

$$\sum_{n=0}^{\infty} \mathcal{H}_{n,\chi}(u; a, b, c; \lambda) \frac{t^n}{n!} \sum_{n=0}^{\infty} (x \ln c)^n \frac{t^n}{n!} = \sum_{n=0}^{\infty} \mathcal{H}_{n,\chi}(x; u; a, b, c; \lambda) \frac{t^n}{n!}.$$

From the above equation, we obtain

$$\sum_{n=0}^{\infty} \left(\sum_{j=0}^n \binom{n}{j} (x \ln c)^{n-j} \mathcal{H}_{j,\chi}(u; a, b, c; \lambda) \right) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \mathcal{H}_{n,\chi}(x; u; a, b, c; \lambda) \frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result. \square

4.3. Recurrence relation for the generalized Eulerian type polynomials. In this section we are going to differentiate (4.1) with respect to the variable t to derive a recurrence relation for the generalized Eulerian type polynomials. Therefore, we obtain the following differential equation:

$$\begin{aligned} \frac{\partial}{\partial t} F_{\lambda}(t, x; u, a, b, c) &= (\ln a) F_{\lambda}(t, x; u, a, b, c) + \frac{\ln a}{t} f_B(x, 1, b, c; \frac{\lambda}{u}; 1) \\ &\quad - \frac{\ln(b^{\lambda})}{ut} F_{\lambda}(t, x; u, a, b, c) f_B(1, 1, b, b; \frac{\lambda}{u}; 1) \\ &\quad + \ln(c^x) F_{\lambda}(t, x; u, a, b, c). \end{aligned}$$

By using this equation, we obtain a recurrence relation for the generalized Eulerian type polynomials by the following theorem:

Theorem 20. *Let $n \in \mathbb{N}$. We have*

$$\begin{aligned} n\mathcal{H}_n(x; u; a, b, c; \lambda) &= (\ln a) \left(n\mathcal{H}_{n-1}(x; u; a, b, c; \lambda) + \mathfrak{B}_n(x; \frac{\lambda}{u}; a, b, c) \right) \\ &\quad - \frac{\lambda \ln b}{u} \sum_{j=0}^n \binom{n}{j} \mathcal{H}_j(x; u; a, b, c; \lambda) \mathfrak{B}_{n-j}(1; \frac{\lambda}{u}; 1, b, b) \\ &\quad + (\ln(c^{nx})) \mathcal{H}_{n-1}(x; u; a, b, c; \lambda), \end{aligned}$$

where $\mathfrak{B}_n(x; \lambda; a, b, c)$ denotes the generalized Bernoulli polynomials of order 1.

Remark 26. *When $a = \lambda = 1$ and $b = c = e$, the recurrence relation for the generalized Eulerian type polynomials is reduced to*

$$nH_n(x; u) = nxH_{n-1}(x; u) - \frac{1}{u} \sum_{j=0}^n \binom{n}{j} H_j(x; u) \mathcal{B}_{n-j}(1; \frac{1}{u}).$$

5. NEW IDENTITIES INVOLVING FAMILIES OF POLYNOMIALS

In this section, we derive some new identities related to the generalized Bernoulli polynomials and numbers of order 1, the Eulerian type polynomials and the generalized array type polynomials.

Theorem 21. *The following relationship holds true:*

$$\mathfrak{B}_n(x; \lambda; a, b, b) = \sum_{j=0}^n \binom{n}{j} \mathcal{H}_j(x; \lambda^{-1}; a, \frac{b}{a}, \frac{b}{a}; 1) \mathfrak{B}_{n-j}(x-1; \lambda; 1, a, a).$$

Proof.

$$\sum_{n=0}^{\infty} \mathfrak{B}_n(x; \lambda; a, b, b) \frac{t^n}{n!} = \left(\frac{ta^{(x-1)t}}{\lambda a^t - 1} \right) \left(\frac{(a^t - \lambda^{-1}) \left(\frac{b}{a} \right)^{xt}}{\left(\frac{b}{a} \right)^t - \lambda^{-1}} \right).$$

Combining (4.17) and (4.1) with the above equation, we get

$$\sum_{n=0}^{\infty} \mathfrak{B}_n(x; \lambda; a, b, b) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \mathfrak{B}_n(x-1; \lambda; 1, a, a) \frac{t^n}{n!} \sum_{n=0}^{\infty} \mathcal{H}_n(x; \lambda^{-1}; a, \frac{b}{a}, \frac{b}{a}; 1) \frac{t^n}{n!}.$$

Therefore

$$\sum_{n=0}^{\infty} \mathfrak{B}_n(x; \lambda; a, b, b) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left(\sum_{j=0}^n \binom{n}{j} \mathcal{H}_j(x; \lambda^{-1}; a, \frac{b}{a}, \frac{b}{a}; 1) \mathfrak{B}_{n-j}(x-1; \lambda; 1, a, a) \right) \frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result. \square

Relationship between the generalized Bernoulli numbers and the Frobenius Euler numbers is given by the following result:

Theorem 22. *The following relationship holds true:*

$$\mathfrak{B}_n(\lambda; a, b) = \frac{1}{\lambda - 1} \sum_{j=0}^n \binom{n}{j} j (\ln a^{-1})^{n-j} \left(\ln \left(\frac{b}{a} \right) \right)^j H_{j-1}(\lambda^{-1}). \quad (5.1)$$

Proof. By using (4.17), we obtain

$$\sum_{n=0}^{\infty} \mathfrak{B}_n(\lambda; a, b) \frac{t^n}{n!} = \frac{ta^{-t}}{\lambda - 1} \left(\frac{1 - \lambda^{-1}}{e^{t \ln(\frac{b}{a})} - \lambda^{-1}} \right).$$

From the above equation, we get

$$\sum_{n=0}^{\infty} \mathfrak{B}_n(\lambda; a, b) \frac{t^n}{n!} = \frac{1}{\lambda - 1} \sum_{n=0}^{\infty} \left(\ln \left(\frac{1}{a} \right) \right)^n \frac{t^n}{n!} \sum_{n=0}^{\infty} n \mathcal{H}_n(\lambda^{-1}) \left(\ln \left(\frac{b}{a} \right) \right)^n \frac{t^n}{n!}.$$

Therefore

$$\sum_{n=0}^{\infty} \mathfrak{B}_n(\lambda; a, b) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left(\sum_{j=0}^n \binom{n}{j} \frac{j (\ln a^{-1})^{n-j} (\ln(\frac{b}{a}))^j}{\lambda - 1} H_{j-1}(\lambda^{-1}) \right) \frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result. \square

Remark 27. *By substituting $a = 1$ and $b = e$ into (5.1), we have*

$$\mathcal{B}_n(\lambda) = \frac{n}{\lambda - 1} H_{n-1}(\lambda^{-1}),$$

cf. [21].

Relationship between the generalized Eulerian type polynomials and generalized array type polynomials are given by the following theorem:

Theorem 23. *The following relationship holds true:*

$$\mathcal{H}_n(x; u; a, b, b; \lambda) = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{d=0}^n \binom{m+k-1}{m} \binom{n}{d} \frac{k! (\ln a^m)^{n-d}}{u^{m+k}} \mathcal{S}_k^d(x; a, b; \lambda).$$

Proof. From (4.1), we obtain

$$\sum_{n=0}^{\infty} \mathcal{H}_n(x; u; a, b, c; \lambda) \frac{t^n}{n!} = \sum_{k=0}^{\infty} \left(\frac{\lambda b^t - a^t}{u - a^t} \right)^k b^{xt}.$$

Combining (3.3) with the above equation, we get

$$\sum_{n=0}^{\infty} \mathcal{H}_n(x; u; a, b, b; \lambda) \frac{t^n}{n!} = \sum_{k=0}^{\infty} \frac{k!}{(u - a^t)^k} \sum_{n=0}^{\infty} \mathcal{S}_k^n(x; a, b; \lambda) \frac{t^n}{n!}.$$

From the above equation, we get

$$\sum_{n=0}^{\infty} \mathcal{H}_n(x; u; a, b, b; \lambda) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{k! \mathcal{S}_k^n(x; a, b; \lambda)}{u^k \left(1 - \frac{a^t}{u} \right)^k} \frac{t^n}{n!}.$$

Now we assume $\left| \frac{a^t}{u} \right| < 1$ in the above equation; thus we get

$$\begin{aligned} & \sum_{n=0}^{\infty} \mathcal{H}_n(x; u; a, b, b; \lambda) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \binom{m+k-1}{m} \frac{k! \mathcal{S}_k^n(x; a, b; \lambda)}{u^{k+m}} \frac{a^{mt} t^n}{n!}. \end{aligned}$$

Therefore

$$\begin{aligned} & \sum_{n=0}^{\infty} \mathcal{H}_n(x; u; a, b, b; \lambda) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{d=0}^n \binom{m+k-1}{m} \binom{n}{d} \frac{k! (\ln a^m)^{n-d}}{u^{m+k}} \mathcal{S}_k^d(x; a, b; \lambda) \right) \frac{t^n}{n!}. \end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result. \square

Remark 28. Substituting $a = 1$ into the above Theorem and noting that $d = n$, we deduce the following identity:

$$\mathcal{H}_n(x; u; 1, b, b; \lambda) = \sum_{k=0}^{\infty} \frac{k!}{(u-1)^k} \mathcal{S}_k^n(x; 1, b; \lambda)$$

which upon setting $\lambda = 1$ and $b = e$, yields

$$H_n(x; u) = \sum_{k=0}^n \frac{k!}{(u-1)^k} \mathcal{S}_k^n(x)$$

which was found by Chang and Ha [12, Lemma 1].

6. RELATIONSHIP BETWEEN THE GENERALIZED BERNOULLI POLYNOMIALS AND THE GENERALIZED ARRAY TYPE POLYNOMIALS

In this section, we give some applications related to the generalized Bernoulli polynomials, generalized array type polynomials. We derive many identities involving these polynomials. By using same method with Agoh and Dilcher's [1], we give the following Theorem:

Theorem 24.

$$\left(\frac{\lambda b^t - a^t}{t} \right)^k b^{xt} = \sum_{n=0}^{\infty} \frac{\mathcal{S}_k^{n+k}(x; a, b; \lambda)}{\binom{n+k}{k}} \frac{t^n}{n!}. \quad (6.1)$$

Proof. Combining (3.3) and (3.2), we get

$$\begin{aligned} \left(\frac{\lambda b^t - a^t}{t} \right)^k b^{xt} &= \frac{1}{t^k} \sum_{n=0}^{\infty} \frac{k!}{n!} \mathcal{S}_k^n(x, a, b; \lambda) t^n \\ &= \sum_{n=0}^{\infty} \frac{k!}{n!} \mathcal{S}_k^{n+k}(x, a, b; \lambda) t^{n-k}. \end{aligned}$$

From the above equation, we arrive at the desired result. \square

Remark 29. By setting $x = 0$, $a = \lambda = 1$ and $b = e$, Theorem 24 yields the corresponding result which is proven by Agoh and Dilcher [1].

Theorem 25.

$$\begin{aligned} & (n+k) \frac{\mathcal{S}_k^{n+k}(x; a, b; \lambda)}{\binom{n+k}{k}} - xn \frac{\mathcal{S}_k^{n+k-1}(x; a, b; \lambda)}{\binom{n+k-1}{k}} \\ &= \sum_{j=0}^n \frac{\binom{n}{j}}{\binom{j+k-1}{k-1}} \mathcal{S}_{k-1}^{j+k-1}(x; a, b; \lambda) \left(\ln(b^{\lambda k}) (\ln(b))^{n-j} - \ln(a^k) (\ln(a))^{n-j} \right). \end{aligned}$$

Proof. By differentiating both sides of equation (6.1) with respect to the variable t , after some elementary calculations, we get the formula asserted by Theorem 25. \square

Theorem 26. The following relationship holds true:

$$\mathcal{S}_{k-1}^n(x+y; a, b; \lambda) = \sum_{j=0}^n \frac{\binom{n}{j} \binom{n+k-1}{k-1}}{\binom{j+k}{k}} \mathcal{S}_k^{j+k}(x; a, b; \lambda) \mathfrak{B}_{n-j}(y; \lambda; a, b, b).$$

Proof. We set

$$\left(\frac{\lambda b^t - a^t}{t} \right)^k b^{xt} \left(\frac{t b^{yt}}{\lambda b^t - a^t} \right) = \left(\frac{\lambda b^t - a^t}{t} \right)^{k-1} b^{(x+y)t}.$$

Combining (6.1) and (4.21) with the above equation, we get

$$\sum_{n=0}^{\infty} \frac{\mathcal{S}_{k-1}^{n+k-1}(x+y; a, b; \lambda) t^n}{\binom{n+k-1}{k-1} n!} = \sum_{n=0}^{\infty} \mathfrak{B}_n(y; \lambda; a, b, b) \frac{t^n}{n!} \sum_{n=0}^{\infty} \frac{\mathcal{S}_k^{n+k}(x; a, b; \lambda) t^n}{\binom{n+k}{k} n!}.$$

Therefore

$$\sum_{n=0}^{\infty} \frac{\mathcal{S}_{k-1}^{n+k-1}(x+y; a, b; \lambda) t^n}{\binom{n+k-1}{k-1} n!} = \sum_{n=0}^{\infty} \left(\sum_{j=0}^n \frac{\binom{n}{j}}{\binom{j+k}{k}} \mathcal{S}_k^{j+k}(x; a, b; \lambda) \mathfrak{B}_{n-j}(y; \lambda; a, b, b) \right) \frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result. \square

Remark 30. By setting $x = y = 0$, $a = \lambda = 1$ and $b = e$, Theorem 26 yields the corresponding result which is proven by Agoh and Dilcher [1].

Theorem 27. The following relationship holds true:

$$\mathfrak{B}_n^{(u-v)}(x+y; \lambda; a, b, b) = \sum_{j=0}^n \frac{\binom{n}{j}}{\binom{n+v}{v}} \mathcal{S}_v^{j+v}(x; a, b; \lambda) \mathfrak{B}_{n-j}^{(u)}(y; \lambda; a, b, b).$$

Proof. We set

$$\left(\frac{\lambda b^t - a^t}{t}\right)^v b^{xt} \left(\frac{t}{\lambda b^t - a^t}\right)^u b^{yt} = \left(\frac{t}{\lambda b^t - a^t}\right)^{u-v} b^{(x+y)t}. \quad (6.2)$$

Combining (6.1) and (4.17) with the above equation, by using same calculations with the proof of Theorem 26, we arrive at the desired result. \square

7. APPLICATION OF THE LAPLACE TRANSFORM TO THE GENERATING FUNCTIONS FOR THE GENERALIZED BERNOULLI POLYNOMIALS AND THE GENERALIZED ARRAY TYPE POLYNOMIALS

In this section, we give an application of the Laplace transform to the generating function for the generalized Bernoulli polynomials and the generalized array type polynomials. We obtain interesting series representation for the families of these polynomials.

By using (6.2), we obtain

$$\begin{aligned} & \sum_{n=0}^{\infty} \mathfrak{B}_n^{(u-v)}(\lambda; a, b, b) \frac{t^n}{n!} e^{-t(y-x) \ln b} \\ &= \sum_{n=0}^{\infty} \left(\sum_{j=0}^n \frac{\binom{n}{j}}{\binom{n+v}{v}} \mathcal{S}_v^{j+v}(x; a, b; \lambda) \mathfrak{B}_{n-j}^{(u)}(\lambda; a, b, b) \right) \frac{t^n}{n!} e^{-ty \ln b}. \end{aligned}$$

Integrate this equation (by parts) with respect to t from 0 to ∞ , we get

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{\mathfrak{B}_n^{(u-v)}(\lambda; a, b, b)}{n!} \int_0^{\infty} t^n e^{-t(y-x) \ln b} dt \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{n!} \sum_{j=0}^n \frac{\binom{n}{j}}{\binom{n+v}{v}} \mathcal{S}_v^{j+v}(x; a, b; \lambda) \mathfrak{B}_{n-j}^{(u)}(\lambda; a, b, b) \right) \int_0^{\infty} t^n e^{-ty \ln b} dt. \end{aligned}$$

By using Laplace transform in the above equation, we arrive at the following Theorem:

Theorem 28. *The following relationship holds true:*

$$\sum_{n=0}^{\infty} \frac{\mathfrak{B}_n^{(u-v)}(\lambda; a, b, b)}{(\ln b^{y-x})^{n+1}} = \sum_{n=0}^{\infty} \sum_{j=0}^n \frac{\binom{n}{j}}{\binom{n+v}{v}} \frac{\mathcal{S}_v^{j+v}(x; a, b; \lambda) \mathfrak{B}_{n-j}^{(u)}(\lambda; a, b, b)}{(\ln b^y)^{n+1}}.$$

Remark 31. *When $a = \lambda = 1$ and $b = e$, Theorem 28 is reduced to the following result:*

$$\sum_{n=0}^{\infty} \frac{B_n^{(u-v)}}{(y-x)^{n+1}} = \sum_{n=0}^{\infty} \sum_{j=0}^n \frac{\binom{n}{j}}{\binom{n+v}{v}} \frac{S_v^{j+v}(x) B_{n-j}^{(u)}}{y^{n+1}}.$$

8. APPLICATIONS THE p -ADIC INTEGRAL TO THE FAMILY OF THE NORMALIZED POLYNOMIALS AND THE GENERALIZED λ -STIRLING TYPE NUMBERS

By using the p -adic integrals on \mathbb{Z}_p , we derive some new identities related to the Bernoulli numbers, the Euler numbers, the generalized Eulerian type numbers and the generalized λ -Stirling type numbers.

In order to prove the main results in this section, we recall each of the following known results related to the p -adic integral.

Let p be a fixed prime. It is known that

$$\mu_q(x + p^N \mathbb{Z}_p) = \frac{q^x}{[p^N]_q}$$

is a distribution on \mathbb{Z}_p for $q \in \mathbb{C}_p$ with $|1 - q|_p < 1$, cf. [19]. Let $UD(\mathbb{Z}_p)$ be the set of uniformly differentiable functions on \mathbb{Z}_p . The p -adic q -integral of the function $f \in UD(\mathbb{Z}_p)$ is defined by Kim [19] as follows:

$$\int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N \rightarrow \infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x) q^x,$$

where

$$[x] = \frac{1 - q^x}{1 - q}.$$

From this equation, the *bosonic* p -adic integral (p -adic Volkenborn integral) was considered from a physical point of view to the bosonic limit $q \rightarrow 1$, as follows ([19]):

$$\int_{\mathbb{Z}_p} f(x) d\mu_1(x) = \lim_{N \rightarrow \infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x), \quad (8.1)$$

where

$$\mu_1(x + p^N \mathbb{Z}_p) = \frac{1}{p^N}.$$

The p -adic q -integral is used in many branch of mathematics, mathematical physics and other areas cf. ([2], [19], [21], [37], [38], [41], [42], [47], [52]).

By using (8.1), we have the Witt's formula for the Bernoulli numbers B_n as follows:

$$\int_{\mathbb{Z}_p} x^n d\mu_1(x) = B_n \quad (8.2)$$

cf. ([2], [19], [20], [22], [37], [52]).

We consider the *fermionic* integral in contrast to the convential bosonic, which is called the fermionic p -adic integral on \mathbb{Z}_p cf. [20]. That is

$$\int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = \lim_{N \rightarrow \infty} \sum_{x=0}^{p^N-1} (-1)^x f(x) \quad (8.3)$$

where

$$\mu_1(x + p^N \mathbb{Z}_p) = \frac{(-1)^x}{p^N}$$

cf. [20]. By using (8.3), we have the Witt's formula for the Euler numbers E_n as follows:

$$\int_{\mathbb{Z}_p} x^n d\mu_{-1}(x) = E_n, \quad (8.4)$$

cf. ([20], [22], [42], [47]).

The Volkenborn integral in terms of the Mahler coefficients is given by the following Theorem:

Theorem 29. *Let*

$$f(x) = \sum_{j=0}^{\infty} a_j \binom{x}{j} \in UD(\mathbb{Z}_p).$$

Then

$$\int_{\mathbb{Z}_p} f(x) d\mu_1(x) = \sum_{j=0}^{\infty} a_j \frac{(-1)^j}{j+1}.$$

Proof of Theorem 29 was given by Schikhof [37].

Theorem 30.

$$\int_{\mathbb{Z}_p} \binom{x}{j} d\mu_1(x) = \frac{(-1)^j}{j+1}.$$

Proof of Theorem 30 was given by Schikhof [37].

Theorem 31. *The following relationship holds true:*

$$B_m = \frac{1}{\ln^m b} \sum_{j=0}^m (-1)^j \frac{j!}{j+1} \mathcal{S}(m, j; 1, b; 1). \quad (8.5)$$

Proof. If we substitute $a = \lambda = 1$ in Theorem 3, we have

$$(\ln b^x)^m = \sum_{j=0}^m \binom{x}{j} j! \mathcal{S}(m, j; 1, b; 1).$$

By applying the p -adic Volkenborn integral with Theorem 30 to the both sides of the above equation, we arrive at the desired result. \square

Remark 32. *By substituting $b = 1$ into (8.5), we have*

$$B_m = \sum_{j=0}^m (-1)^j \frac{j!}{j+1} S(m, j)$$

where $S(m, j)$ denotes the Stirling numbers of the second kind cf. ([11], [15], [23]).

Theorem 32. *The following relationship holds true:*

$$\begin{aligned} & \sum_{j=0}^n \binom{n}{j} (\ln a)^{n-j} (\ln c)^j B_j - u(\ln c)^n B_n \\ &= \sum_{j=0}^n \binom{n}{j} (\ln c)^j \left(\lambda (\mathcal{H}(u; a, b, c; \lambda) + \ln b)^{n-j} - u\mathcal{H}_{n-j}(u; a, b, c; \lambda) \right) B_j. \end{aligned}$$

Proof. By using Theorem 10, we have

$$\begin{aligned} & \sum_{j=0}^n \binom{n}{j} (\ln a)^{n-j} (\ln c)^j x^j - u(\ln c)^n x^n \\ &= \sum_{j=0}^n \binom{n}{j} (\ln c)^j x^j \left(\lambda (\mathcal{H}(u; a, b, c; \lambda) + \ln b)^{n-j} - u\mathcal{H}_{n-j}(u; a, b, c; \lambda) \right). \end{aligned} \tag{8.6}$$

By applying Volkenborn integral in (8.1) to the both sides of the above equation, we get

$$\begin{aligned} & \sum_{j=0}^n \binom{n}{j} (\ln a)^{n-j} (\ln c)^j \int_{\mathbb{Z}_p} x^j d\mu(x) - u(\ln c)^n \int_{\mathbb{Z}_p} x^n d\mu(x) \\ &= \sum_{j=0}^n \binom{n}{j} (\ln c)^j \left(\lambda (\mathcal{H}(u; a, b, c; \lambda) + \ln b)^{n-j} - u\mathcal{H}_{n-j}(u; a, b, c; \lambda) \right) \int_{\mathbb{Z}_p} x^j d\mu(x). \end{aligned}$$

By substituting (8.2) into the above equation, we easily arrive at the desired result. \square

Remark 33. *By substituting $b = c = e$ and $a = \lambda = 1$ into Theorem 32, we arrive at the following nice identity:*

$$B_n = \frac{1}{1-u} \sum_{j=0}^n \binom{n}{j} \left((H(u) + 1)^{n-j} - uH_{n-j}(u) \right) B_j.$$

Theorem 33. *The following relationship holds true:*

$$\begin{aligned} & \sum_{j=0}^n \binom{n}{j} (\ln a)^{n-j} (\ln c)^j E_j - u(\ln c)^n E_n \\ &= \sum_{j=0}^n \binom{n}{j} (\ln c)^j \left(\lambda (\mathcal{H}(u; a, b, c; \lambda) + \ln b)^{n-j} - u\mathcal{H}_{n-j}(u; a, b, c; \lambda) \right) E_j. \end{aligned}$$

Proof. Proof of Theorem 33 is same as that of Theorem 32. Combining (8.3), (8.6) and (8.4), we easily arrive at the desired result. \square

Remark 34. *By substituting $b = c = e$ and $a = \lambda = 1$ into Theorem 33, we arrive at the following nice identity:*

$$E_n = \frac{1}{1-u} \sum_{j=0}^n \binom{n}{j} \left((H(u) + 1)^{n-j} - uH_{n-j}(u) \right) E_j.$$

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